Take the moments of momenta about P; $\therefore\ ua\sqrt{2}-va+k^2w=0$.

Substitute the values of u and k^2 in this equation, $\therefore\ aw=\frac{3}{8}v$, $\therefore\ v=v.\frac{3}{8}\sqrt{2}$.

The impulse f which must be combined with the initial momentum v in order that the cube may have the resultant momentum u is

$$f^2=v^2+u^2-2vu\cos\tfrac{1}{4}\pi=(1+\tfrac{18}{64}-\tfrac{6}{8})v^2;\ \therefore f=v.\tfrac{1}{8}\sqrt{34}.$$

To find the inclination a, we have $u^2=v^2+f^2-2vf\cos a$;

$$\therefore\ \cos a=5\div\sqrt{34},$$

which is *constant.*

That this is the correct result may be seen by showing that f applied at P in the direction a will cause the angular velocity w found above. For take moments about C; $k^2w=fa\sqrt{2}.\sin(\frac{1}{4}\pi-a)=fa(\cos a-\sin a)$;

$$\therefore\ \tfrac{2}{3}aw=\frac{\sqrt{34}}{8}v\left(\frac{5-3}{\sqrt{34}}\right)=\frac{v}{4};\ \therefore\ aw=\tfrac{3}{8}v,$$

which is the result previously obtained.

The initial velocity of the cube will be just sufficient to overturn it when the energy of translation and rotation are sufficient to raise the center C to a point directly above P; i. e., when

$$u^2+k^2w=2ga\sqrt{2}.[1-\cos(\tfrac{1}{4}\pi-\beta)];$$
$$\therefore\quad v^2=\tfrac{16}{3}\sqrt{2}.ag[1-\cos(\tfrac{1}{4}\pi-\beta)].$$

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

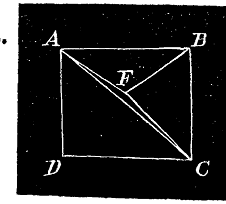
Solutions of problems in No. 4 have been received as follows:

From R. J. Adcock, 358; Prof. W. P. Casey, 355, 356, 358; George Eastwood, 356, 358; Prof. Asaph Hall, 358; Prof. E. B. Seitz, 355, 358; R. S. Woodward, 358. We also received a solution of 353, of No. 3, from the proposer, Mr. Hoover, and a solution of 352 from Mr. Heaton.

355. "The length of a garden, in the form of a parallelogram, is one rod greater than the breadth. Within the garden is a fountain; and a gravel walk extends diagonally across the garden, from corner to corner, and the distance from the fountain to one end of said walk is three rods, and to the other end four rods; and from this end of the walk along one end of the garden, to the next corner, and from thence to the fountain, is eight rods. Required the area of the garden."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.

Let $ABCD$ be the rectangular garden, AC the diagonal walk, and F the fountain.



Let $BC = x$, $AB = x+1$, $FA = 3$ rods, $FC = 4$ rods. Then since $CB + BF = 8$ rods, $BF = 8 - x$;

$$\cos ABF = \frac{(x+1)^2 + (8-x)^2 - 3^2}{2(x+1)(x-8)} = \frac{x^2 - 7x + 28}{(x+1)(x-8)},$$

$$\cos CBF = \frac{x^2 + (8-x)^2 - 4^2}{2x(8-x)} = \frac{x^2 - 8x + 24}{x(8-x)}.$$

But $\cos ABF = \sin CBF$, whence we find $\cos^2 ABF + \cos^2 CBF = 1$.

By substitution and reduction we find

$$x^6 - 14x^5 + 153x^4 - 680x^3 + 640x^2 + 768x + 576 = 0.$$

This equation has two positive roots, $x = 3$, and $x = 4.4236748$. For the first value of x the sides of the garden would be 3 and 4 rods, and the fountain would be situated at D, the corner of the garden. For the second value of x the fountain is situated as represented in the diagram, and the area of the garden $= x(x+1) = 23.99257$ square rods.

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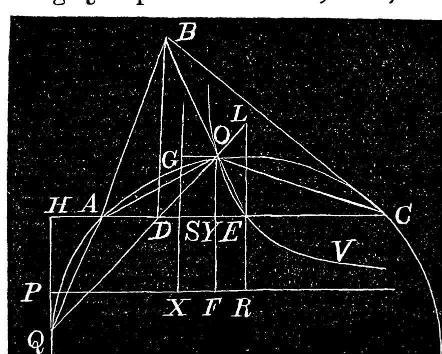
356. "In a triangle ABC, BD is perpendicular to the base AC, and O is the center of gravity of the triangle. Join AO, DO and CO. Given the base AC and the angles AOD, AOC to construct the triangle ABC."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Analysis. Look upon the line AC as being given in position, therefore the points A, C are given, and as the angle AOC is given, therefore the circle $QAOC$ is given. Produce OD to Q, and as the angle AOQ is given, therefore the line AQ is given, and, as A is a given point, therefore Q is a given point. Draw BO and produce it to E and E is a given point, then draw EL perpendicular to AC meeting QO produced in L; and, BO being equal to $2OE$, $\therefore$ $DO = 2OL$. The perpendicular QH to CA produced is in position, and $\therefore$ QH is a given line. And as $DL : LO$ so make $HE : ES$; $\therefore$ ES is a given line and S a given point; and as $LD : DO$ so make $QH : HP$; $\therefore$ HP is a given line and P a given point; through P, O and S, draw PR, OG parallel to AC, and OF, GH par'l to BD.



Now $HE : ES :: DL : LO :: DE : EY$; $\therefore HD : SY :: DE : EY$, (i. e.) $ED : DH :: EY : YS$; (1) and, again, $QH : HP :: LD : DO :: LE : OY$; $\therefore LE : QH :: OY : HP$ or $YF :: ED : DH :: EY : YS$, by (1); $\therefore EY : YS :: OY : YF$, and (Euc. VI., 16) the fig. YG = fig. YR; to each add the fig. YX, $\therefore$ the rectangle OX = rectangle EX; but the rectangle EX is given, whence the locus of O is a rectangular hyperbola OEV whose asymptotes are XG, XR. Therefore O is a given p't; whence B is a given point, and $\triangle ABC$ is given. The const'n is evident.

COR. If through a given point a line be drawn intersecting two given lines, and if the portion of this line intercepted by the two given ones be divided in a given ratio, the locus of the point of section is a hyperbola.

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357. [No solution received. A solution by the proposer is invited]

358. "Given the angles A, B and C of a plane triangle, and $d \log a$, $d \log b$ and $d \log c$; a, b, c being the sides respectively.

What are the corresponding values of dA, dB and dC expressed in seconds of arc?"

SOLUTION BY R. S. WOODWARD, DETROIT, MICH.

Since in any plane triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

$d \log \sin A - d \log a = d \log \sin B - d \log b = d \log \sin C - d \log c.$

Put α = log sin dif. for $1''$ corresponing to A,

β = " " " " " " " B,

γ = " " " " " " " C.

Then, dA, dB and dC being expressed in seconds of arc, the above equation becomes

$$\alpha dA - d \log a = \beta dB - d \log b = \gamma dC - d \log c = Q \text{ say.} \qquad (1)$$

Since $A + B + C = 180°$,

$$dA + dB + dC = 0. \qquad (2)$$

From (1)

$$dA = \frac{Q + d \log a}{\alpha},$$

$$dB = \frac{Q + d \log b}{\beta}, \qquad (3)$$

$$dC = \frac{Q + d \log c}{\gamma},$$

which substituted in (2) gives

$$Q = -\frac{\beta\gamma\, d\log a + \gamma\alpha\, d\log b + \alpha\beta\, d\log c}{\alpha\beta + \alpha\gamma + \beta\gamma}.$$

Therefore by (3)

$$dA = \frac{\beta(d\log a - d\log c) + \gamma(d\log a - d\log b)}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

$$dB = \frac{\gamma(d\log b - d\log a) + \alpha(d\log b - d\log c}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

$$dC = \frac{\alpha(d\log c - d\log b) + \beta(d\log c - d\log a)}{\alpha\beta + \alpha\gamma + \beta\gamma}.$$

SOLUTION BY PROF. ASAPH HALL, NAVAL OBSERVAT'Y, WASH., D. C.

Projecting the sides b and c of a plane triangle on the side a we have

$$a = b\cos C + c\cos B;$$

and in a similar manner we find two more equations of this kind. Differentiating these, considering all the parts variable and noticing the condition

$$A + B + C = \pi = \text{constant},$$

we have the three symmetrical differential equations of a plane triangle,

$$da = \cos C.db + \cos B.dc + c\sin B.dA,$$
$$db = \cos A.dc + \cos C.da + a\sin C.dB,$$
$$dc = \cos B.da + \cos A.db + b\sin A.dC.$$

The quantities $\cos C.db$ and $\cos B.dc$ are the increments of the sides b and c projected on the side a; and the sum of these applied to da gives the total increment of the side a. Also $c\sin B$ is the perpendicular from the angle A on the opposite side, and the total increment of a divided by c sin B gives dA. For seconds of are we must multiply this ratio by 206264.8, the number of seconds in radius. Since

$$da = \frac{a.da}{a} = a.d\log a,$$

we may use the differentials themselves or $d\log a$, &c. One should notice the similarity of these differential equations to those of spherical trigonometry.

SOLUTION OF PART (2), PROB. 344, BY PROF. SEITZ (SEE P. 101).—Let P be the third random point, and EF the random chord, through it.

Draw the radius OL perpendicular to EF. Let p_0, p_1, p_2, p_3 be the respective probabilities that AB, CD, EF will intersect in 0, 1, 2, 3 points.

Let $EP = z$, $EF = z'$, $\angle EOL = \psi$, and $\angle KOL = \rho$. Then $z' = 2r \times \sin\psi$; an element of the circle at P is $r\sin\psi\, d\psi\, dz$, and for an elemental change in the direction of EF we have $d\rho$.